

SCHATTEN PROPERTIES OF TOEPLITZ OPERATORS ON THE PALEY-WIENER SPACE

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ABSTRACT. We collect several old and new descriptions of Schatten class Toeplitz operators on the Paley-Wiener space and answer a question on discrete Hilbert transform commutators posed by Richard Rochberg.

1. INTRODUCTION

Given a bounded function φ on the real line, \mathbb{R} , consider the Toeplitz operator T_φ on the classical Paley-Wiener space PW_a ,

$$T_\varphi: f \mapsto P_a(\varphi f), \quad f \in PW_a. \quad (1)$$

The space PW_a could be regarded as the subspace in $L^2(\mathbb{R})$ of functions with Fourier spectrum in the interval $[-a, a]$, symbol P_a above denotes the orthogonal projection in $L^2(\mathbb{R})$ to PW_a . Basic theory of Toeplitz operators on PW_a can be found in paper [9] by R. Rochberg.

We are interested in description of Schatten class Toeplitz operators on PW_a in terms of their standard symbols. By the standard symbol of an operator in (1) we mean the entire function $\varphi_{st} = \mathcal{F}^{-1}\chi_{2a}\mathcal{F}\varphi$, where \mathcal{F} denotes the Fourier transform on the Schwartz space of tempered distributions, and χ_{2a} is the indicator function of the interval $(-2a, 2a)$. As we will see, a Toeplitz operator T_φ on PW_a belongs to the Schatten class \mathcal{S}^p , $0 < p < \infty$, if and only if $e^{2iax}\varphi_{st}$ belongs to a discrete oscillation Besov space introduced in 1987 by R. Rochberg [9]. Its definition we now recall.

For a measure μ on \mathbb{R} and a function $f \in L^1_{\text{loc}}(\mu)$, the oscillation of order n of f on an interval $I \subset \mathbb{R}$ with respect to μ is defined by

$$\text{osc}(f, I, \mu, n) = \inf_{P_n} \frac{1}{\mu(I)} \int_I |f(x) - P_n(x)| d\mu(x),$$

where the infimum is taken over all polynomials P_n of degree at most n . If $\mu(I) = 0$, we put $\text{osc}_I(f, I, \mu, n) = 0$. Define the family \mathcal{I}_a of closed intervals

$$I_{a,j,k} = \left[\frac{2\pi}{a} k 2^j, \frac{2\pi}{a} (k+1) 2^j \right], \quad j, k \in \mathbb{Z}, \quad j \geq 0.$$

Note that endpoints of intervals in \mathcal{I}_a belong to the lattice $\mathbb{Z}_a = \left\{ \frac{2\pi}{a} k, k \in \mathbb{Z} \right\}$. Let p be a positive real number, and let $\left[\frac{1}{p} \right]$ be the integer part of $\frac{1}{p}$. The discrete

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oscillation Besov space $\mathbb{B}_p(a, \text{osc}) = \mathbb{B}_{p,p}^{1/p}(\mathbb{Z}_a, \mu_a, \text{osc})$ is defined by

$$\mathbb{B}_p(a, \text{osc}) = \left\{ f \in L_{\text{loc}}^1(\mu_a) : \|f\|_{\mathbb{B}_p(a, \text{osc})}^p = \sum_{I \in \mathcal{I}_a} \text{osc} \left(f, I, \mu_a, \left[\frac{1}{p} \right] \right)^p < \infty \right\},$$

where $\mu_a = \frac{2\pi}{a} \sum_{x \in \mathbb{Z}_a} \delta_x$ is the normalized counting measure on \mathbb{Z}_a .

Our main result is the following theorem.

Theorem 1. *Let a, p be positive real numbers, let φ be a bounded function on \mathbb{R} , and let φ_{st} be the standard symbol of the Toeplitz operator T_φ on PW_a . Then we have $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$ if and only if $e^{2iax}\varphi_{st} \in \mathbb{B}_p(4a, \text{osc})$. Moreover, $\|T_\varphi\|_{\mathcal{S}^p}$ is comparable to $\|e^{2iax}\varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})}$ with constants depending only on p .*

Theorem 1 complements a classical description of Toeplitz operators in $\mathcal{S}^p(\text{PW}_a)$ given by R. Rochberg [9] for $1 \leq p < \infty$ and extended by V. Peller [5] to the whole range $0 < p < \infty$. To formulate the result, consider a system $\{\nu_j\}_{j \leq -1}$ of infinitely smooth functions on \mathbb{R} such that

$$\text{supp } \nu_j \subset [2^{j-1}, 2^j], \quad 0 \leq \nu_j \leq 1, \quad \nu_{j-1}(x) = \nu_j(x/2), \quad \sum \nu_j = 1 \text{ on } (0, \frac{1}{3}].$$

Define $\nu_j(x) = \nu_{-j}(1-x)$ for real $x \geq \frac{1}{2}$ and integer $j \geq 1$, put $\nu_0 = 1 - \sum_{j \neq 0} \nu_j$ for $j = 0$. Finally, let $\nu_{a,j}(x) = \nu_j((x+a)/2a)$ for all $x \in [-a, a]$ and $j \in \mathbb{Z}$. Observe that system $\{\nu_{a,j}\}$ provides a resolution of unity on the interval $[-a, a]$ by functions supported on subintervals I_j whose lengths are comparable to the distance from I_j to the endpoints of $[-a, a]$. Rochberg-Peller theorem says that T_φ is in $\mathcal{S}^p(\text{PW}_a)$ for $0 < p < \infty$ if and only if

$$a \sum_{j \in \mathbb{Z}} 2^{-|j|} \cdot \|\mathcal{F}^{-1}(\nu_{2a,j} \cdot \mathcal{F}\varphi)\|_{L^p(\mathbb{R})}^p < \infty,$$

with control of the norms. R. Rochberg gives yet another characterization of Toeplitz operators in class $\mathcal{S}^p(\text{PW}_a)$, $1 \leq p < \infty$, in terms of a reproducing kernel decomposition of their standard symbols, see Theorem 5.3 in [9]. Both the statement and the proof of his result for $p = 1$ contain errors that we correct in Section 3.

As a consequence of Theorem 1, we obtain the following result.

Theorem 2. *Let $a > 0$. The discrete Hilbert transform commutator*

$$C_\psi : f \mapsto \frac{1}{\pi} \oint_{\mathbb{Z}_a} \frac{\psi(x) - \psi(t)}{x - t} f(t) d\mu_a(t), \quad f \in L^2(\mu_a),$$

belongs to the trace class $\mathcal{S}^1(L^2(\mu_a))$ if and only if $\psi \in \mathbb{B}_1(a, \text{osc}) \cap L^\infty(\mathbb{Z}_a)$.

This answers the question posed by R. Rochberg in 1987. See Section 6 for a summary of results on discrete Hilbert transform commutators and an analogue of Theorem 2 for the case $0 < p < 1$.

We would like to mention papers [11], [12] by R. Torres for readers interested in wavelet characterizations and interpolation theory of discrete Besov spaces. The problem of membership in Schatten classes \mathcal{S}^p for general truncated Toeplitz operators has been recently studied by P. Lopatto and R. Rochberg [3], see also Section 4.3 in author's paper [1].

2. PROOF OF THEOREM 1 FOR $1 < p < \infty$

Theorem 1 for $1 < p < \infty$ follows from known results. Let $\mathbb{B}_p(\mathbb{R}) = \dot{\mathbb{B}}_{p,p}^{1/p}(\mathbb{R})$ be the standard homogeneous Besov space on the real line \mathbb{R} , see, e.g., Chapter 3 in [4] for definition and basic properties. Given a Toeplitz operator T_φ on PW_a with symbol $\varphi \in L^\infty(\mathbb{R})$, we denote

$$\varphi_{st}^- = \mathcal{F}^{-1} \chi_{(-2a,0)} \mathcal{F} \varphi, \quad \varphi_{st}^+ = \mathcal{F}^{-1} \chi_{[0,2a)} \mathcal{F} \varphi,$$

where χ_S is the indicator function of a set S . As usual, \mathcal{F} stands for the Fourier transform on the Schwartz space of tempered distributions. The following result is a combination of Theorem 5.1 and its Corollary in [9].

Theorem (R. Rochberg). *Let $1 < p < \infty$ and let $a > 0$. Then a Toeplitz operator T_φ on PW_a belongs to $\mathcal{S}_p(\text{PW}_a)$ if and only if $\|e^{2iax} \varphi_{st}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|e^{-2iax} \varphi_{st}^+\|_{\mathbb{B}_p(\mathbb{R})}$ is finite, in which case $\|T_\varphi\|_{\mathcal{S}^p}$ is comparable to $\|e^{2iax} \varphi_{st}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|e^{-2iax} \varphi_{st}^+\|_{\mathbb{B}_p(\mathbb{R})}$ with constants depending only on p .*

Denote by \mathcal{E}_a the set of tempered distributions whose Fourier transforms are supported on the interval $[-a, a]$. Next result is Theorem 1 in [12].

Theorem (R. Torres). *Let $1 < p < \infty$ and let f be a function in $\mathcal{E}_a \cap \mathbb{B}_p(\mathbb{R})$ for some $a > 0$. Then its restriction to \mathbb{Z}_{2a} belongs to $\mathbb{B}_p(2a, \text{osc})$ and $\|f\|_{\mathbb{B}_p(2a, \text{osc})}$ is comparable to $\|f\|_{\mathbb{B}_p(\mathbb{R})}$ with constants depending only on p . Moreover, every sequence in $\mathbb{B}_p(a, \text{osc})$ is the restriction to \mathbb{Z}_a of a unique function (modulo polynomials) in $\mathcal{E}_a \cap \mathbb{B}_p(\mathbb{R})$.*

Proof of Theorem 1 ($1 < p < \infty$). Let φ be a bounded function of \mathbb{R} and let $\varphi_{st} = \mathcal{F}^{-1} \chi_{(-2a,2a)} \mathcal{F} \varphi$ be the standard symbol of the Toeplitz operator $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$. Then functions $e^{2iax} \varphi_{st}^-$, $e^{-2iax} \varphi_{st}^+$ belong to $\mathcal{E}_{2a} \cap \mathbb{B}_p(\mathbb{R})$ by R. Rochberg's theorem above. From theorem by R. Torres we see that $e^{2iax} \varphi_{st}^- \in \mathbb{B}_p(4a, \text{osc})$ and $e^{-2iax} \varphi_{st}^+ \in \mathbb{B}_p(4a, \text{osc})$ with control of the norms. Now observe that $e^{4iax} = 1$ and $e^{2iax} \varphi_{st} = e^{2iax} \varphi_{st}^- + e^{-2iax} \varphi_{st}^+$ on \mathbb{Z}_{4a} , hence $e^{2iax} \varphi_{st} \in \mathbb{B}_p(4a, \text{osc})$.

Conversely, assume that the restriction of $e^{2iax} \varphi_{st}$ to \mathbb{Z}_{4a} is in $\mathbb{B}_p(4a, \text{osc})$. Using theorem by R. Torres, find a function $f \in \mathcal{E}_{2a} \cap \mathbb{B}_p(\mathbb{R})$ such that its restriction to \mathbb{Z}_{4a} agrees with $e^{2iax} \varphi_{st}$. Put $f^- = \mathcal{F}^{-1} \chi_{(-2a,0)} \mathcal{F} f$ and $f^+ = \mathcal{F}^{-1} \chi_{[0,2a)} \mathcal{F} f$. Observe that $\tilde{\varphi} = e^{-2iax} f^+ + e^{2iax} f^-$ is an entire function of exponential type at most $2a$ coinciding with φ_{st} on \mathbb{Z}_{4a} . Since φ_{st} , $\tilde{\varphi}$ are the first order distributions supported on the finite interval $[-2a, 2a]$, we have $|\tilde{\varphi}(x)| + |\varphi(x)| \leq c + c|x|$ for all $x \in \mathbb{R}$ and a constant $c \geq 0$. It follows that the entire function $\frac{\tilde{\varphi} - \varphi}{z}$ of exponential type at most $2a$ is bounded on \mathbb{R} and vanishes on $\mathbb{Z}_{4a} \setminus \{0\}$, hence $\tilde{\varphi} - \varphi_{st} = p \sin(2az)$ for a polynomial p of degree at most 1. Therefore, we have $T_\varphi = T_{\varphi_{st}} = T_{\tilde{\varphi}}$ on PW_a , see Section 2.D in [9]. Since $f^\pm \in \mathbb{B}_p(\mathbb{R})$, we can use R. Rochberg's theorem and conclude that $T_{\tilde{\varphi}} \in \mathcal{S}^p(\text{PW}_a)$ with control of the norms: $\|T_{\tilde{\varphi}}\|_{\mathcal{S}^p}$ is controllable by $\|e^{2iax} \tilde{\varphi}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|e^{-2iax} \tilde{\varphi}^+\|_{\mathbb{B}_p(\mathbb{R})} \leq c_p \|f\|_{\mathbb{B}_p(\mathbb{R})} \leq \tilde{c}_p \|e^{2iax} \varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})}$. \square

3. REPRODUCING KERNEL DECOMPOSITION OF STANDARD SYMBOLS

In this section we show that the standard symbol of a Toeplitz operator on PW_a from class \mathcal{S}^p could be represented as a linear combination of normalized reproducing kernels of PW_{2a} with coefficients c_k such that $\sum |c_k|^p < \infty$. We consider only the case $0 < p \leq 1$. Proposition 3.1 below is a corrected version of

Theorem 5.3 in [9]. In the original statement the author of [9] forgot to normalize the exponentials in formula (5.6) of [9]. More importantly, he used the fact that the Fourier multiplier $f \mapsto \mathcal{F}^{-1} \chi_{[0,1]} \mathcal{F} f$ is bounded on $\mathbb{B}_p(\mathbb{R})$. This is not the case for $p = 1$. Here is a more accurate implementation of the ideas from [9].

Let ψ be a bounded function on the real line \mathbb{R} . Consider the standard Hardy space H^2 in the upper half-plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \operatorname{Im} \lambda > 0\}$ of the complex plane \mathbb{C} . Denote by H_-^2 the anti-analytic subspace $\{f \in L^2(\mathbb{R}) : \bar{f} \in H^2\}$ of $L^2(\mathbb{R})$. Recall that the classical Hankel operator $H_\psi : H^2 \rightarrow H_-^2$ is defined by

$$H_\psi : f \mapsto P_-(\psi f), \quad f \in H^2,$$

where P_- denotes the orthogonal projection from $L^2(\mathbb{R})$ to H_-^2 . The operator H_ψ is completely determined by its standard anti-analytic symbol $\psi_{st} = \mathcal{F}^{-1} \chi_{(-\infty, 0)} \mathcal{F} \psi$. The latter means that $H_\psi f = H_{\psi_{st}} f$ for all $f \in H^2$ such that $\sup_{x \in \mathbb{R}} |x f(x)| < \infty$. Take a positive number $\varepsilon > 0$ and define the sets $\mathcal{U}_\varepsilon^+$, $\mathcal{U}_\varepsilon^-$ by

$$\mathcal{U}_\varepsilon^\pm = \{\lambda \in \mathbb{C} : \lambda = (1 + \varepsilon)^m (\varepsilon x \pm i); \quad x, m \in \mathbb{Z}\}.$$

For $\lambda \in \mathbb{C}^+$ let us denote by k_λ the reproducing kernel of H^2 at λ , $k_\lambda = -\frac{1}{2\pi i} \frac{1}{z - \lambda}$.

Theorem (R. Rochberg [8]). *There exists a number $\varepsilon > 0$ such that $H_\psi \in \mathcal{S}^p(H^2)$ if and only if $\psi_{st} = \sum_{\lambda \in \mathcal{U}_\varepsilon^+} c_\lambda \frac{\bar{k}_\lambda}{\|k_\lambda\|^2}$, where $\sum |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of ψ_{st} in this form is comparable to $\|H_\psi\|_{\mathcal{S}^p}^p$ with constants depending only on $p \in (0, \infty)$.*

Remark that for $p \in (0, 1]$ the series defining ψ_{st} in the theorem above converges absolutely to a bounded function on \mathbb{R} , while for $p > 1$ the convergence holds only in the Besov space $\mathbb{B}_p(\mathbb{R})$ (one need to extract constant terms from every summand to get the convergent series, see discussion in [8]). In order to prove an analogous result for Toeplitz operators on the Paley-Wiener space, let us consider the sets

$$\mathcal{U}_{\eta a, \varepsilon}^\pm = \{\lambda \in \mathcal{U}_\varepsilon^\pm : |\operatorname{Im} \lambda| > \frac{\varepsilon}{\eta a}\}, \quad \Lambda_{\eta a, \varepsilon} = \mathcal{U}_{\eta a, \varepsilon}^- \cup \mathbb{Z}_{\eta a} \cup \mathcal{U}_{\eta a, \varepsilon}^+.$$

Here $\mathbb{Z}_{\eta a} = \{\frac{2\pi}{\eta a} k, \quad k \in \mathbb{Z}\}$. Next, for $a > 0$ and $\lambda \in \mathbb{C}$, denote by $\rho_{a, \lambda}$ the reproducing kernel of the space PW_a at the point λ . Recall that

$$\rho_{a, \lambda} : z \mapsto \frac{1}{\pi} \frac{\sin a(z - \bar{\lambda})}{z - \bar{\lambda}}, \quad z \in \mathbb{C}.$$

We are going to prove the following proposition.

Proposition 3.1. *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist $\varepsilon > 0$, $\eta > 1$ such that $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$ if and only if $\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$, where $\sum_\lambda |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of φ_{st} in this form is comparable to $\|T_\varphi\|_{\mathcal{S}^p}^p$ with constants depending only on $p \in (0, 1]$.*

We will show how to reduce Proposition 3.1 to the above theorem for Hankel operators using a splitting of the standard symbol into three pieces: analytic, anti-analytic and a piece with “small” Fourier support.

The following two results for $0 < p \leq 1$ are consequences of Lemma 1 and Lemma 2 from [5]. The range $1 \leq p < \infty$ has been treated earlier in [9], see also Section 2 in [10].

Lemma 3.1. *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist bounded functions φ_ℓ , φ_c , and φ_r such that $T_\varphi = T_{\varphi_\ell} + T_{\varphi_c} + T_{\varphi_r}$ on PW_a ,*

$$\text{supp } \mathcal{F}\varphi_\ell \subset [-4a, -\frac{a}{2}], \quad \text{supp } \mathcal{F}\varphi_c \subset [-a, a], \quad \text{supp } \mathcal{F}\varphi_r \subset [\frac{a}{2}, 4a],$$

and we have $\|T_{\varphi_s}\|_{\mathcal{S}^p} \leq c_p \|T_\varphi\|_{\mathcal{S}^p}$ for every $s = \ell, c, r$ for $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$. Here c_p is a constant depending only on p .

Lemma 3.2. *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$ be such that $\text{supp } \hat{\varphi} \subset [-a, a]$. Then we have $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$ if and only if $\varphi \in L^p(\mathbb{R})$, in which case $\|\varphi\|_{L^p(\mathbb{R})}$ is comparable to $\|T_\varphi\|_{\mathcal{S}^p}$ with constants depending only on p .*

Proof of Proposition 3.1. Let $\varphi \in L^\infty(\mathbb{R})$ and let $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a, 2a)}\mathcal{F}\varphi$ be the standard symbol of the operator T_φ on PW_a . Then $T_\varphi = T_{\varphi_{st}}$, see Section 2.D in [9]. Suppose that $\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$ for some $\varepsilon > 0$, $\eta > 0$, and some coefficients c_λ such that $\sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p < \infty$. It follows from the estimate

$$\frac{|\rho_{2a, \lambda}(z)|}{\|\rho_{a, \lambda}\|^2} \leq ce^{2a|\text{Im } z|}, \quad z \in \mathbb{C}, \quad \lambda \in \mathbb{C},$$

that this series converges absolutely to an entire function of exponential type at most $2a$ bounded on the real line \mathbb{R} . By triangle inequality (see, e.g., Theorem A1.1 in [6]), we have

$$\|T_\varphi\|_{\mathcal{S}^p}^p = \|T_{\varphi_{st}}\|_{\mathcal{S}^p}^p \leq \left(\sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p \right) \sup_{\lambda \in \mathbb{C}} \|T_{\varphi_\lambda}\|_{\mathcal{S}^p}^p,$$

where we denoted $\varphi_\lambda = \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$. Take $\lambda \in \mathbb{C}$. For every $f, g \in \text{PW}_a$ we have

$$(T_{\rho_{2a, \lambda}} f, g) = (f \bar{g}, \rho_{2a, \bar{\lambda}}) = f(\bar{\lambda}) \cdot \overline{g(\bar{\lambda})} = (f, \rho_{a, \bar{\lambda}})(\rho_{a, \lambda}, g).$$

It follows that the operator T_{φ_λ} has rank one and $\|T_{\varphi_\lambda}\|_{\mathcal{S}^p} = 1$. Hence T_φ belongs to $\mathcal{S}^p(\text{PW}_a)$ and $\|T_\varphi\|_{\mathcal{S}^p}^p \leq \sum_{\lambda} |c_\lambda|^p$.

Now let φ be a bounded function on \mathbb{R} such that $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$. We want to show that the standard symbol $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a, 2a)}\mathcal{F}\varphi$ of T_φ can be represented in the form

$$\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$$

for some positive numbers ε, η depending only on p and a sequence $\{c_\lambda\}$ such that $\sum_{\lambda} |c_\lambda|^p$ is comparable to $\|T_\varphi\|_{\mathcal{S}^p}^p$. By Lemma 3.1, it suffices to consider separately the following three cases: (1) $\text{supp } \hat{\varphi} \subset (-\infty, 0]$; (2) $\text{supp } \hat{\varphi} \subset [-a, a]$; (3) $\text{supp } \hat{\varphi} \subset [0, +\infty)$. Let us treat the third case first. Denote by $M_{e^{-iax}}$ the operator of multiplication by e^{-iax} on $L^2(\mathbb{R})$. Since $\text{supp } \hat{\varphi} \subset [0, +\infty)$, we have

$$H_{e^{-2iax}\varphi} = M_{e^{-iax}} T_\varphi P_a M_{e^{-iax}},$$

where $H_{e^{-2iax}\varphi} : H^2 \rightarrow H^2$ is the Hankel operator with symbol $\psi = e^{-2iax}\varphi$. In particular, we have $\|H_\psi\|_{\mathcal{S}^p} \leq \|T_\varphi\|_{\mathcal{S}^p}$. By Rochberg's Theorem above, the anti-analytic function $\psi_{st} = \mathcal{F}^{-1}\chi_{(-\infty, 0)}\mathcal{F}e^{-2iax}\varphi$ admits the following representation:

$$\psi_{st} = \sum_{\lambda \in \mathcal{U}_\varepsilon^+} c_\lambda \frac{\bar{k}_\lambda}{\|k_\lambda\|^2},$$

where $\sum_{\lambda \in \mathcal{U}_\varepsilon^+} |c_\lambda|^p$ is comparable to $\|H_\psi\|_{\mathcal{S}^p}^p$, and $\varepsilon > 0$ does not depend on ψ . This gives us decomposition for φ_{st} :

$$\varphi_{st} = e^{2iax} \psi_{st} = \sum_{\lambda \in \mathcal{U}_\varepsilon^+} c_\lambda \frac{e^{2iax} \overline{k_\lambda}}{\|k_\lambda\|^2} = \sum_{\lambda \in \mathcal{U}_\varepsilon^+} c_\lambda \frac{P_{2a}(e^{2iax} \overline{k_\lambda})}{\|k_\lambda\|^2},$$

where P_{2a} denotes the orthogonal projection in $L^2(\mathbb{R})$ to PW_{2a} . It is easy to see that $P_{2a}(e^{2iax} \overline{k_\lambda}) = e^{2ia\bar{\lambda}} \rho_{2a, \bar{\lambda}}$ and $\|\rho_{a, \bar{\lambda}}\|^2 \leq 2e^{2a \text{Im } \lambda} \cdot \|k_\lambda\|_{L^2(\mathbb{R})}^2$, hence

$$\varphi_{st} = \sum_{\lambda \in \mathcal{U}_\varepsilon^-} c_{\bar{\lambda}} \beta_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}$$

for some complex numbers β_λ such that $\sup_\lambda |\beta_\lambda| \leq 2$. Next, in the case where $\text{supp } \varphi \subset (-\infty, 0]$ we can consider the adjoint operator $T_\varphi^* = T_{\varphi_{st}^*}$ with the standard symbol $\varphi_{st}^* : z \mapsto \overline{\varphi_{st}(\bar{z})}$ and conclude that in this situation

$$\varphi_{st} = \sum_{\lambda \in \mathcal{U}_\varepsilon^+} \overline{c_\lambda \beta_{\bar{\lambda}}} \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}.$$

Now let $\text{supp } \varphi \subset [-a, a]$. By Lemma 3.2, we have $\varphi \in L^p(\mathbb{R})$. In particular, $\varphi \in \text{PW}_{2a}$ and Plancherel-Polya theorem [7] yields the following decomposition:

$$\varphi = \varphi_{st} = \frac{\pi}{2a} \sum_{\lambda \in \mathbb{Z}_{2a}} f(\lambda) \rho_{2a, \lambda}, \quad \sum_{\lambda \in \mathbb{Z}_{2a}} |f(\lambda)|^p \leq c_p a^p \|\varphi\|_{L^p(\mathbb{R})}^p,$$

where the constant c_p depends only on p . Put $\Lambda_\varepsilon = \mathcal{U}_\varepsilon^+ \cup \mathbb{Z}_{2a} \cup \mathcal{U}_\varepsilon^-$. To summarize, we have proved that for every bounded function φ on \mathbb{R} such that $T_\varphi \in \mathcal{S}^p(\text{PW}_a)$ there are coefficients c_λ , $\lambda \in \Lambda_\varepsilon$, such that

$$\varphi_{st} = \sum_{\lambda \in \Lambda_\varepsilon} c_\lambda \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}, \quad \sum_{\lambda \in \Lambda_\varepsilon} |c_\lambda|^p \leq c_p \|T_\varphi\|_{\mathcal{S}^p}^p. \quad (2)$$

It remains to show that the set Λ_ε and coefficients c_λ in this decomposition could be replaced by the set $\Lambda_{\eta a, \varepsilon}$ and some new coefficients c_λ satisfying the second estimate in (2). To this end, for every point $\lambda \in \Lambda_\varepsilon$ denote by ζ_λ the nearest point to λ in $\Lambda_{\eta a, \varepsilon} \subset \Lambda_\varepsilon$, where $\eta = 2^k$ and $k \in \mathbb{Z}$ is a positive integer number that will be specified later. Consider the function

$$\tilde{\varphi}^{(1)} = \sum_{\lambda \in \Lambda_\varepsilon} c_\lambda \frac{\rho_{2a, \zeta_\lambda}}{\|\rho_{a, \zeta_\lambda}\|^2} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} \tilde{c}_\lambda^{(1)} \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}, \quad \tilde{c}_\lambda^{(1)} = \sum_{\nu \in \Lambda_\varepsilon, \zeta_\nu = \lambda} c_\nu$$

Note that $\tilde{\varphi}^{(1)}$ has the required representation and $\sum |\tilde{c}_\lambda^{(1)}|^p \leq \sum |c_\lambda|^p$. Moreover, we have $\|T_\varphi - T_{\tilde{\varphi}^{(1)}}\|_{\mathcal{S}^p}^p \leq \sum_{\lambda \in \Lambda_\varepsilon \setminus \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p \cdot \|T_{\varphi_\lambda} - T_{\varphi_{\zeta_\lambda}}\|_{\mathcal{S}^p}^p$. On the other hand, the quasi-norm in \mathcal{S}_p of the rank two operator

$$T_{\varphi_\lambda} - T_{\varphi_{\zeta_\lambda}} = \frac{\rho_{a, \lambda}}{\|\rho_{a, \lambda}\|} \otimes \frac{\rho_{a, \lambda}}{\|\rho_{a, \lambda}\|} - \frac{\rho_{a, \zeta_\lambda}}{\|\rho_{a, \zeta_\lambda}\|} \otimes \frac{\rho_{a, \zeta_\lambda}}{\|\rho_{a, \zeta_\lambda}\|}$$

does not exceed

$$2^{\frac{1}{p}} \left\| \frac{\rho_{a, \zeta_\lambda}}{\|\rho_{a, \zeta_\lambda}\|} - \frac{\rho_{a, \lambda}}{\|\rho_{a, \lambda}\|} \right\|_{L^2(\mathbb{R})} \leq 2^{\frac{1}{p} + \frac{1}{2}} \left(1 - \frac{\text{Re } \rho_{a, \zeta_\lambda}(\lambda)}{\|\rho_{a, \zeta_\lambda}\| \cdot \|\rho_{a, \lambda}\|} \right)^{\frac{1}{2}}.$$

Since $|\zeta_\lambda - \lambda| \leq \frac{2\pi}{\eta a}$ for all λ by construction, one can choose a large number $\eta = 2^k$ so that $\|T_\varphi - T_{\tilde{\varphi}^{(1)}}\|_{\mathcal{S}^p}^p \leq \frac{1}{2} \|T_\varphi\|_{\mathcal{S}^p}^p$. Clearly, this choice of η does not depend on φ

and a . Iterating the process, we see that there are functions

$$\tilde{\varphi}^{(n)} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} \tilde{c}_\lambda^{(n)} \frac{\rho_{2a, \lambda}}{\|\rho_{a, \lambda}\|^2}, \quad n = 1, 2, \dots,$$

such that $\|T_\varphi - T_{\tilde{\varphi}^{(1)}} - \dots - T_{\tilde{\varphi}^{(n)}}\|_{\mathcal{S}^p}^p \leq \frac{1}{2^n} \|T_\varphi\|_{\mathcal{S}^p}^p$ and $\sum_{n, \lambda} |\tilde{c}_\lambda^{(n)}|^p \leq c_p^p \|T_\varphi\|_{\mathcal{S}^p}^p$. Since $\mathcal{S}^p(\text{PW}_a)$ is a complete quasi-normed space and a Toeplitz operator on PW_a is zero if and only if its standard symbol is zero (see Section 2.D in [9]), this gives us the required decomposition of φ_{st} with coefficients $c_\lambda = \sum_{n \geq 1} \tilde{c}_\lambda^{(n)}$, $\lambda \in \Lambda_{\eta a, \varepsilon}$.

4. INTERPOLATION OF DISCRETE BESOV SEQUENCES

Denote by $\text{PW}_{[0, a]}$ the Paley-Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[0, a]$. Recall that the reproducing kernel $k_{a, \lambda}$ of the space $\text{PW}_{[0, a]}$ at a point $\lambda \in \mathbb{C}_+$ has the form

$$k_{a, \lambda}(z) = -\frac{1}{2\pi i} \frac{1 - e^{ia(z - \bar{\lambda})}}{z - \bar{\lambda}}, \quad z \in \mathbb{C}.$$

Denote by $\mathcal{C}_0(\mathbb{Z}_a)$ the set of functions on \mathbb{Z}_a tending to zero at infinity. Our aim in this section is to prove the following proposition.

Proposition 4.1. *Let $0 < p \leq 1$, let Λ be the set $\Lambda_{\eta a, \varepsilon}$ from Proposition 3.1, and let $F = \sum_{\lambda \in \Lambda} c_\lambda \frac{k_{a, \lambda}}{\|k_{\frac{a}{2}, \lambda}\|^2}$ for some $c_\lambda \in \mathbb{C}$ such that $\sum_{\lambda \in \Lambda} |c_\lambda|^p < \infty$. Then the restriction of F to \mathbb{Z}_a belongs to $\mathbb{B}_p(a, \text{osc}) \cap \mathcal{C}_0(\mathbb{Z}_a)$. Conversely, for every function $f \in \mathbb{B}_p(a, \text{osc})$ there exists the unique function F as above and a polynomial q of degree at most $\lceil \frac{1}{p} \rceil$ such that $f = q + F$ on \mathbb{Z}_a . Moreover, the infimum of $\sum_{\lambda \in \Lambda} |c_\lambda|^p$ over all possible representations of $F = \sum_{\lambda \in \Lambda} c_\lambda \frac{k_{a, \lambda}}{\|k_{\frac{a}{2}, \lambda}\|^2}$ in this form is comparable to $\|f\|_{\mathbb{B}_p(a, \text{osc})}^p$ with constants depending only on p .*

The proof of Proposition 4.1 is based on the following lemma.

Lemma 4.1. *We have $\|k_{a, \lambda}\|_{\mathbb{B}_p(a, \text{osc})} \leq c_p \|k_{\frac{a}{2}, \lambda}\|^2$ for every $a > 0$, $0 < p \leq 1$, and $\lambda \in \mathbb{C}$, where the constant c_p depends only on p .*

Proof. At first, consider the points λ in the support of μ_a . For $\lambda \in \mathbb{Z}_a$ we have

$$k_{a, \lambda}(x) = \begin{cases} \|k_{a, \lambda}\|^2, & x = \lambda; \\ 0, & x \in \text{supp } \mu_a \setminus \{\lambda\}. \end{cases}$$

Taking $P_I = 0$ for all intervals $I \in \mathcal{I}_a$ in the definition of $\text{osc}(k_{a, \lambda}, I, \mu_a, \lceil \frac{1}{p} \rceil)$, we obtain the estimate

$$\begin{aligned} \|k_{a, \lambda}\|_{\mathbb{B}_p(a, \text{osc})}^p &\leq \sum_{I \in \mathcal{I}_a} \left(\frac{1}{\mu_a(I)} \int_I |k_{a, \lambda}(x)| d\mu_a(x) \right)^p \\ &= \|k_{a, \lambda}\|^{2p} \mu_a(\{\lambda\})^p \sum_{I \in \mathcal{I}_a} \frac{\chi_I(\lambda)}{\mu_a(I)^p} \\ &\leq c_p \|k_{\frac{a}{2}, \lambda}\|^{2p}. \end{aligned}$$

Now let λ be an arbitrary point in $\mathbb{C} \setminus \text{supp } \mu_a$. Then $k_{a,\lambda}(x) = -\frac{1}{2\pi i} \frac{1-e^{-ia\bar{\lambda}}}{x-\lambda}$ for all $x \in \text{supp } \mu_a$. Thus, we need to estimate an oscillation of the function $x \mapsto \frac{1}{x-\bar{\lambda}}$ on the lattice \mathbb{Z}_a . Divide collection \mathcal{I}_a from Section 1 into two parts:

$$\mathcal{I}_{a,1} = \{I \in \mathcal{I}_a : I = I_{a,j,k}, \text{Re } \lambda \notin I_{a,j,k-1} \cup I_{a,j,k} \cup I_{a,j,k+1}\}, \quad \mathcal{I}_{a,2} = \mathcal{I}_a \setminus \mathcal{I}_{a,1}.$$

For an interval $I \in \mathcal{I}_{a,1}$ with center x_c , define the polynomial P_I of degree $[\frac{1}{p}]$ by

$$\frac{1}{x-\bar{\lambda}} - P_I(x) = \frac{(x-x_c)^{[\frac{1}{p}]+1}}{(x-\bar{\lambda})(\bar{\lambda}-x_c)^{[\frac{1}{p}]+1}}. \quad (3)$$

Using this polynomial, we can estimate

$$\text{osc} \left(\frac{1}{x-\bar{\lambda}}, I, \mu_a, [\frac{1}{p}] \right) \leq \sup_{x \in I} \left| \frac{(x-x_c)^{[\frac{1}{p}]+1}}{(x-\bar{\lambda})(\bar{\lambda}-x_c)^{[\frac{1}{p}]+1}} \right| \leq \frac{|I|^{[\frac{1}{p}]+1}}{\text{dist}(\lambda, I)^{[\frac{1}{p}]+2}}, \quad (4)$$

where $|I|$ denotes the length of I . Since $I \in \mathcal{I}_{a,1}$, we have $\text{dist}(\lambda, I) \geq |I|$, hence

$$\sum_{I \in \mathcal{I}_{a,1}} \text{osc} \left(\frac{1}{\bar{\lambda}-x}, I, \mu_a, [\frac{1}{p}] \right)^p \leq \sum_{I \in \mathcal{I}_{a,1}} \frac{1}{|I|^p} \leq c_p \cdot a^p. \quad (5)$$

We also will need a more accurate estimate for the left hand side of the inequality above in the case where $|\text{Im } \lambda|$ is large. For every $j \geq 0$, let $\mathcal{I}_{a,1}^j$ be the set of intervals $I_{a,j,k}$, $k \in \mathbb{Z}$, belonging to the family $\mathcal{I}_{a,1}$. We have

$$\begin{aligned} \sum_{I \in \mathcal{I}_{a,1}^j} \left(\frac{|I|^{[\frac{1}{p}]+1}}{\text{dist}(\lambda, I)^{[\frac{1}{p}]+2}} \right)^p &= \sum_{I \in \mathcal{I}_{a,1}^j} \left(\frac{|I|^{[\frac{1}{p}]+1}}{(|\text{Im } \lambda|^2 + \text{dist}(\text{Re } \lambda, I)^2)^{([\frac{1}{p}]+2)/2}} \right)^p \\ &\leq c_p \left(\frac{a}{2^j} \right)^p \sum_{m \geq 1} \left(\frac{1}{(\frac{a}{2^j})^2 |\text{Im } \lambda|^2 + m^2} \right)^{\frac{1}{2}[\frac{1}{p}]p+p} \\ &\leq c_p \left(\frac{a}{2^j} \right)^p \gamma_j^{1-[\frac{1}{p}]p-2p}, \end{aligned}$$

where $\gamma_j = \max(1, \frac{a}{2^j} |\text{Im } \lambda|)$. Indeed, the last inequality follows from elementary estimates

$$\sum_{m=1}^{\infty} m^{-1-2p} < \infty, \quad \int_1^{\infty} \frac{dx}{(r^2+x^2)^s} \leq c_s r^{1-2s},$$

where $r > 0$, and the constant c_s depends on $s > 1/2$. Put $N_\lambda = [\log_2(a|\text{Im } \lambda|)]$ if $a|\text{Im } \lambda| \geq 2$ and $N_\lambda = 0$ otherwise. Note that $\tilde{p} = -1 + [\frac{1}{p}]p + p$ is a positive number. It follows

$$\begin{aligned} \sum_{I \in \mathcal{I}_{a,1}} \text{osc} \left(\frac{1}{\bar{\lambda}-x}, I, \mu_a, [\frac{1}{p}] \right)^p &\leq c_p \sum_{j=0}^{\infty} \left(\frac{a}{2^j} \right)^p \gamma_j^{1-[\frac{1}{p}]p-2p} \\ &\leq c_p a^{-\tilde{p}} |\text{Im } \lambda|^{-\tilde{p}-p} \sum_{j=0}^{N_\lambda} 2^{\tilde{p}j} + c_p \sum_{j=N_\lambda}^{\infty} \frac{a^p}{2^{pj}} \\ &\leq \frac{c_p}{|\text{Im } \lambda|^{\tilde{p}}}. \end{aligned}$$

Combining the last estimate with (5), we get

$$\sum_{I \in \mathcal{I}_{a,1}} \text{osc} \left(\frac{1}{\bar{\lambda} - x}, I, \mu_a, \left[\frac{1}{p} \right] \right)^p \leq c_p \min \left(a^p, \frac{1}{|\text{Im } \lambda|^p} \right).$$

Now consider the family $\mathcal{I}_{a,2} = \mathcal{I}_{a,21} \cup \mathcal{I}_{a,22}$,

$$\mathcal{I}_{a,21} = \{I \in \mathcal{I}_{a,2} : |I| \leq |\text{Im } \lambda|\}, \quad \mathcal{I}_{a,22} = \{I \in \mathcal{I}_{a,2} : |I| > |\text{Im } \lambda|\}.$$

For an interval $I \in \mathcal{I}_{a,21}$ we use the polynomial P_I defined by (3). Then formula (4) implies

$$\sum_{I \in \mathcal{I}_{a,21}} \text{osc} \left(\frac{1}{\bar{\lambda} - x}, I, \mu_a, \left[\frac{1}{p} \right] \right)^p \leq \sum_{I \in \mathcal{I}_{a,21}} \left(\frac{|I|^{\lfloor \frac{1}{p} \rfloor + 1}}{|\text{Im } \lambda|^{\lfloor \frac{1}{p} \rfloor + 2}} \right)^p \leq \frac{c_p}{|\text{Im } \lambda|^p}.$$

Note that if $|\text{Im } \lambda| < \frac{2\pi}{a}$, the set $\mathcal{I}_{a,21}$ is empty. This shows that we can write

$$\sum_{I \in \mathcal{I}_{a,21}} \text{osc} \left(\frac{1}{\bar{\lambda} - x}, I, \mu_a, \left[\frac{1}{p} \right] \right)^p \leq c_p \min \left(a^p, \frac{1}{|\text{Im } \lambda|^p} \right).$$

For $I \in \mathcal{I}_{a,22}$ we put $P_I = 0$. Denote by x_0 the nearest point to λ in $\text{supp } \mu_a$, and set $I' = I \setminus \{x \in \mathbb{R} : |x - \text{Re } \lambda| < \pi/a\}$. We have

$$\begin{aligned} \frac{1}{\mu_a(I)} \int_I \left| \frac{1}{x - \bar{\lambda}} \right| d\mu_a(x) &\leq \frac{\mu_a(\{x_0\})}{\mu_a(I)|x_0 - \bar{\lambda}|} + \frac{1}{\mu_a(I)} \int_{I'} \frac{dx}{|x - \bar{\lambda}|} \\ &\leq \frac{c}{a|I||x_0 - \bar{\lambda}|} + \frac{c}{|I|} \int_{\pi a^{-1}}^{|I|} \frac{dx}{\sqrt{x^2 + |\text{Im } \lambda|^2}} \\ &\leq \frac{c}{a|I||x_0 - \bar{\lambda}|} + \frac{c}{|I|} \min \left(\log \frac{a|I|}{\pi}, \log \left(\frac{|I|}{|\text{Im } \lambda|} + 1 \right) \right). \end{aligned}$$

Using estimates

$$\sum_{I \in \mathcal{I}_{a,2}} \frac{1}{|I|^p} \leq c_p a^p, \quad \sum_{I \in \mathcal{I}_{a,2}} \left(\frac{\log a|I|}{|I|} \right)^p \leq c_p a^p, \quad \sum_{I \in \mathcal{I}_{a,22}} \left(\frac{1}{|I|} \log \frac{|I|}{|\text{Im } \lambda|} \right)^p \leq \frac{c_p}{|\text{Im } \lambda|^p},$$

we see that

$$\sum_{I \in \mathcal{I}_{a,22}} \text{osc} \left(\frac{c_p}{\bar{\lambda} - x}, I, \mu_a, \left[\frac{1}{p} \right] \right)^p \leq \frac{c_p}{|x_0 - \bar{\lambda}|^p} + c_p \min \left(a^p, \frac{1}{|\text{Im } \lambda|^p} \right).$$

Eventually, we obtain

$$\left\| \frac{1}{x - \bar{\lambda}} \right\|_{\mathbb{B}_p(a, \text{osc})}^p \leq \frac{c_p}{|x_0 - \bar{\lambda}|^p} + c_p \min \left(a^p, \frac{1}{|\text{Im } \lambda|^p} \right).$$

It follows that

$$\|k_{a,\lambda}\|_{\mathbb{B}_p(a, \text{osc})}^p \leq c_p (1 + e^{-a|\text{Im } \lambda|})^p \min \left(a^p, \frac{1}{|\text{Im } \lambda|^p} \right) + c_p \left| \frac{1 - e^{-ia\bar{\lambda}}}{x_0 - \lambda} \right|^p \leq c_p \|k_{\frac{a}{2}, \lambda}\|^{2p},$$

which is the desired estimate. \square

Let $\mathcal{C}_0(\mathbb{R})$ denote the set of all continuous functions on \mathbb{R} tending to zero at infinity. For completeness, we include the proof of the following known lemma.

Lemma 4.2. *Let $0 < p \leq 1$, $a > 0$. For every function $f \in \mathbb{B}_p(\text{osc}, a)$ there exists a function $F \in \mathbb{B}_p(\mathbb{R})$ such that $F = f$ on \mathbb{Z}_a , and*

$$\|F\|_{\mathbb{B}_p(\mathbb{R})} \leq c_p \|f\|_{\mathbb{B}_p(\text{osc}, a)},$$

where the constant c_p depends only p .

Proof. For $k \in \mathbb{Z}$ put $I_k = \left[\frac{2\pi}{a} \left[\frac{1}{p} \right] k, \frac{2\pi}{a} \left[\frac{1}{p} \right] (k+1) \right]$. Interiors of intervals I_k are disjoint and every set $I_k \cap \mathbb{Z}_a$ contains $\left[\frac{1}{p} \right] + 1$ points. On every I_k define the polynomial P_k of degree at most $\left[\frac{1}{p} \right]$ such that $P_k(x) = f(x)$ for all $x \in I_k \cap \mathbb{Z}_a$. Next, set $F(x) = P_k(x)$ for $x \in I_k$. We claim that the function F is in $\mathbb{B}_p(\mathbb{R})$. To check this, let us take an interval $J_{j,k} = \left[\frac{2\pi}{a} \left[\frac{1}{p} \right] k \cdot 2^j, \frac{2\pi}{a} \left[\frac{1}{p} \right] (k+1) \cdot 2^j \right]$ with $k, j \in \mathbb{Z}$. In the case where $j < 0$ we clearly have $\text{osc}(F, J_{j,k}, m, \left[\frac{1}{p} \right]) = 0$ because the function F is a polynomial of degree at most $\left[\frac{1}{p} \right]$ on I . Hence, we can assume that $J = J_{j,k} = I_\ell \cup \dots \cup I_{\ell+N}$ for some $\ell \in \mathbb{Z}$ and $N \geq 1$. Consider the polynomial P_J of degree at most $\left[\frac{1}{p} \right]$ such that

$$\text{osc} \left(f, J, \mu_a, \left[\frac{1}{p} \right] \right) = \frac{1}{\mu_a(J)} \int_J |f(x) - P_J(x)| d\mu_a(x).$$

We have

$$\begin{aligned} \frac{1}{|J|} \int_J |F(x) - P_J(x)| dx &= \frac{1}{|J|} \sum_{s=0}^N \int_{I_{\ell+s}} |P_{\ell+s}(x) - P_J(x)| dx \leq \\ &\leq \frac{c_p}{|J|} \sum_{s=0}^N \int_{I_{\ell+s}} |P_{\ell+s}(x) - P_J(x)| d\mu_a(x) \leq c_p \text{osc} \left(f, I, \mu_a, \left[\frac{1}{p} \right] \right), \end{aligned}$$

where we used the fact that

$$\int_{I_\ell} |P(x)| dx \leq c_p \int_{I_\ell} |P(x)| d\mu_a(x)$$

for every interval I_ℓ , $\ell \in \mathbb{Z}$, and every polynomial P of degree at most $\left[\frac{1}{p} \right]$. It follows that

$$\|F\|_{\mathbb{B}_p(\mathbb{R}, m, \text{osc})}^p \leq c_p^p \sum_{j,k} \text{osc} \left(f, J_{j,k}, \mu_a, \left[\frac{1}{p} \right] \right)^p \leq c_p^p \|f\|_{\mathbb{B}_p(\text{osc}, a)}^p,$$

and hence F belongs to the space $\mathbb{B}_{p,p}^{1/p}(\mathbb{R}, dx, \text{osc}) = \mathbb{B}_p(\mathbb{R})$, as required. \square

Proof of Proposition 4.1. Consider a function F of the form

$$F = \sum_{\lambda \in \Lambda} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}, \quad \sum_{\lambda \in \Lambda} |c_\lambda|^p < \infty.$$

Since $0 < p \leq 1$ and $|k_{a,\lambda}(x)| \leq c \|k_{\frac{a}{2},\lambda}\|^2$ for every $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the series above converges absolutely to a function from $\mathcal{C}_0(\mathbb{R})$ by the Lebesgue dominated convergence theorem. By Lemma 4.1, the restriction of F to \mathbb{Z}_a (to be denoted by f) is in $\mathbb{B}_p(a, \text{osc})$ and $\|f\|_{\mathbb{B}_p(a, \text{osc})}^p \leq c_p \sum_{\lambda \in \Lambda} |c_\lambda|^p$ for a constant c_p depending only on p .

Conversely, take $f \in \mathbb{B}_p(a, \text{osc})$ and find a function $\tilde{F} \in \mathbb{B}_p(\mathbb{R})$ such that $\tilde{F} = f$ on \mathbb{Z}_a , see Lemma 4.2. Applying Theorem 2.10 from [8] to analytic and anti-analytic parts of \tilde{F} , we obtain the representation

$$\tilde{F} = q - \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{U}_\varepsilon} \tilde{c}_\lambda \frac{\text{Im } \lambda}{x - \lambda}, \quad x \in \mathbb{R},$$

where the coefficients $\tilde{c}_k \in \mathbb{C}$ are such that $\sum |\tilde{c}_\lambda|^p \leq c_p \|\tilde{F}\|_{\mathbb{B}_p(\mathbb{R})}^p$, and q is a polynomial of degree at most $[\frac{1}{p}]$. Now consider the function

$$F = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda \frac{k_{\lambda,a}}{\|k_{\frac{a}{2},\lambda}\|^2}, \quad c_\lambda = \tilde{c}_\lambda \frac{\text{Im } \lambda \cdot \|k_{\frac{a}{2},\lambda}\|^2}{1 - e^{-ia\lambda}}.$$

Observe that $|c_\lambda| \leq |\tilde{c}_\lambda|$ for all $\lambda \in \mathcal{U}_\varepsilon$ and $f = q + F$ on \mathbb{Z}_a . We need to replace the set \mathcal{U}_ε above to the set $\Lambda_{\eta a, \varepsilon}$ from Proposition 3.1. Since $k_{\frac{a}{2},\lambda} = e^{\frac{iax}{4}} e^{-\frac{ia\lambda}{4}} \rho_{\frac{a}{4},\lambda}$, we have $\|k_{\frac{a}{2},\lambda}\|^2 = e^{-\frac{a \text{Im } \lambda}{2}} \|\rho_{\frac{a}{4},\lambda}\|^2$ and

$$e^{-\frac{iax}{2}} F = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda e^{-\frac{ia\lambda}{2}} \frac{\rho_{a/2,\lambda}}{\|k_{a,\lambda}\|^2} = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda e^{-\frac{ia \text{Re } \lambda}{2}} \frac{\rho_{a/2,\lambda}}{\|\rho_{a/4,\lambda}\|^2}.$$

From the beginning of the proof of Proposition 3.1 we see that the Toeplitz operator on $\text{PW}_{a/4}$ with symbol $e^{-\frac{iax}{2}} F$ belongs to the class $\mathcal{S}^p(\text{PW}_{a/4})$. It follows that

$$e^{-\frac{iax}{2}} F = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} d_\lambda \frac{\rho_{a/2,\lambda}}{\|\rho_{a/4,\lambda}\|^2}, \quad \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |d_\lambda|^p \leq c_p \sum_{\lambda \in \mathcal{U}_\varepsilon} |c_\lambda|^p.$$

This yields the required representation for F ,

$$F = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}, \quad \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p \leq c_p \|f\|_{\mathbb{B}_p(a, \text{osc})}^p,$$

with some new coefficients c_λ . Since $\sum_\lambda |c_\lambda| < \infty$, the function $G = e^{-\frac{iax}{2}} F$ is an entire function of exponential type at most $a/2$ such that $\lim_{x \rightarrow \pm\infty} |G(x)| = 0$. In particular, it is uniquely determined by values on \mathbb{Z}_a . This proves uniqueness in Proposition 4.1. \square

5. PROOF OF THEOREM 1 FOR $0 < p \leq 1$

Proof of Theorem 1 ($0 < p \leq 1$). Let $\varphi \in L^\infty(\mathbb{R})$ be a function on \mathbb{R} such that the operator T_φ is in $\mathcal{S}^p(\text{PW}_a)$, and let $\varphi_{st} = \mathcal{F}^{-1} \chi_{(-2a, 2a)} \mathcal{F} \varphi$ be the standard symbol of T_φ . By Proposition 3.1 and Proposition 4.1, we have $e^{2iax} \varphi_{st} \in \mathbb{B}_p(4a, \text{osc})$ and moreover, $\|e^{2iax} \varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})} \leq c_p \|T_\varphi\|_{\mathcal{S}^p}$ for a constant c_p depending only on p .

Conversely, assume that the restriction of the function $e^{2iax} \varphi_{st}$ to \mathbb{Z}_{4a} belongs to the space $\mathbb{B}_p(4a, \text{osc})$. By Proposition 4.1, there exists a function F and a polynomial q of degree at most $[\frac{1}{p}]$ such that $q + F = e^{2iax} \varphi_{st}$ on \mathbb{Z}_{4a} and

$$F = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{k_{4a,\lambda}}{\|k_{2a,\lambda}\|^2} = e^{2iax} \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda e^{-2ia \text{Re } \lambda} \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2} \quad (6)$$

for some $c_\lambda \in \mathbb{C}$ such that $\sum |c_\lambda|^p \leq c_p \|e^{2iax} \varphi_{st}\|_{\mathbb{B}_p(4a, \text{osc})}^p$. We claim that $T_{\tilde{\varphi}} = T_\varphi$ on PW_a , where $\tilde{\varphi} = e^{-2iax}(q + F)$. Indeed, the entire function $z \mapsto \tilde{\varphi} - \varphi_{st}$ has exponential type at most $2a$, vanishes on \mathbb{Z}_{4a} , and satisfies a polynomial estimate

on \mathbb{R} . Hence $\tilde{\varphi} - \varphi_{st} = \tilde{q} \sin(2az)$ for all $z \in \mathbb{C}$ and a polynomial \tilde{q} . Thus, we have $T_\varphi = T_{\varphi_{st}} = T_{\tilde{\varphi}}$. It remains to use formula (6) and Proposition 3.1. The theorem is proved. \square

6. DISCRETE HILBERT TRANSFORM COMMUTATORS. PROOF OF THEOREM 2

Recall that $\mu_a = \frac{2\pi}{a} \sum_{x \in \mathbb{Z}_a} \delta_x$ is the scalar multiple of the counting measure on the lattice $\mathbb{Z}_a = \{\frac{2\pi}{a}k, k \in \mathbb{Z}\}$. The discrete Hilbert transform H_{μ_a} on $L^2(\mu_a)$ is defined by

$$H_{\mu_a} : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{f(t)}{x-t} d\mu_a(t),$$

and its commutator $C_\psi = M_\psi H_{\mu_a} - H_{\mu_a} M_\psi$ with the multiplication operator $M_\psi : f \mapsto \psi f$ on $L^2(\mu_a)$ by

$$C_\psi : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{\psi(x) - \psi(t)}{x-t} f(t) d\mu_a(t), \quad x \in \text{supp } \mu_a.$$

It is well-known that the operator H_{μ_a} admits the bounded extension from the dense subset \mathcal{G} of $L^2(\mu_a)$ of finitely supported bounded functions to the whole space $L^2(\mu_a)$. A possible way to define the operator C_ψ on $L^2(\mu_a)$ for any symbol ψ on \mathbb{Z}_a is to consider its bilinear form on elements from the dense subset $\mathcal{G} \times \mathcal{G}$ of $L^2(\mu_a) \times L^2(\mu_a)$. We will also deal with the operators $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ defined by

$$\tilde{C}_\psi : f \mapsto \frac{1}{\pi} \int_{\mathbb{Z}_a} \frac{\psi(x) - \psi(t)}{x-t} f(t) d\mu_{\frac{a}{2}}(t), \quad x \in \text{supp } \nu_{\frac{a}{2}},$$

where the measure $\nu_{\frac{a}{2}} = \frac{4\pi}{a} \sum_{x \in \mathbb{Z}_{\frac{a}{2}}} \delta_{x+\frac{2\pi}{a}}$ is supported on the lattice $\frac{2\pi}{a} + \mathbb{Z}_{\frac{a}{2}}$. It can be shown that for $1 \leq p \leq \infty$ the operator $C_\psi : L^2(\mu_a) \rightarrow L^2(\mu_a)$ is in \mathcal{S}^p if and only if the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ is in \mathcal{S}^p . As we will see, for $0 < p < 1$ we may have $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$ for a function ψ on \mathbb{Z}_a such that the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ is in \mathcal{S}^p .

The discrete Hilbert transform commutators were investigated in details in paper [9]. In particular, it was proved in [9] that C_ψ is bounded on $L^2(\mu_a)$ if and only if its symbol ψ belongs to the discrete BMO(\mathbb{Z}_a) space of functions f on \mathbb{Z}_a such that $\sup_{I \in \mathcal{I}_a} \text{osc}(f, I, \mu_a, 0) < \infty$, where $\mathcal{I}_a = \{I_{a,j,k}, j, k \in \mathbb{Z}, j \geq 0\}$ is the collection of intervals defined in Section 1. Another result from [9] says that C_ψ is compact on $L^2(\mu_a)$ if and only if $\psi \in \text{CMO}(\mathbb{Z}_a)$, that is, $\lim_{k \rightarrow \pm\infty} \text{osc}(\psi, I_{a,j,k}, \mu_a, 0) = 0$ for every $j \geq 0$ and $\lim_{j \rightarrow +\infty} \text{osc}(\psi, J_j, \mu_a, 0) = 0$ for any sequence of intervals $J_j \subset \mathbb{R}$ of length j with common center. Finally, the operator C_ψ belongs to $\mathcal{S}^p(L^2(\mathbb{Z}_a))$ for $1 < p < \infty$ if and only if $\psi \in \mathbb{B}_p(a, \text{osc})$, moreover, we have $C_\psi \in \mathcal{S}^1(L^2(\mu_a))$ for every $\psi \in \mathbb{B}_1(a, \text{osc})$. See Theorem 6.2 in [9] and Theorem 4 in [12] for the proof of these results. It was an open question stated in Section 7 of [9] whether $C_\psi \in \mathcal{S}^p(L^2(\mu_a))$ is equivalent to $\psi \in \mathbb{B}_p(a, \text{osc})$ for all positive p (in particular, for $p = 1$). Theorem 2 gives the affirmative answer to this question for $p = 1$. On the other hand, for $0 < p < 1$ we show that there exists symbols $\psi \in \mathbb{B}_p(a, \text{osc})$ such that $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$. In fact, the following modification of Theorem 2 holds true.

Theorem 2'. *Let $0 < p \leq 1$. The operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ belongs to the class \mathcal{S}^p if and only if $\psi \in \mathbb{B}_p(a, \text{osc}) \cap L^\infty(\mathbb{Z}_a)$. Moreover, the quasi-norms $\|\tilde{C}_\psi\|_{\mathcal{S}^p}$ and $\|\psi\|_{\mathbb{B}_p(a, \text{osc})}$ are comparable with constants depending only on p .*

For the proof we need a result on unitary equivalence of discrete Hilbert transform commutators to some truncated Hankel operators. Given a positive number $a > 0$, we denote by $\text{PW}_{[-a, 0]}$ the Paley-Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[-a, 0]$. Define the truncated Hankel operator $\Gamma_\psi : \text{PW}_{[0, a]} \rightarrow \text{PW}_{[-a, 0]}$ with symbol $\psi \in L^\infty(\mathbb{R})$ by

$$\Gamma_\psi : f \mapsto P_{[-a, 0]}(\psi f), \quad f \in \text{PW}_{[0, a]},$$

where $P_{[-a, 0]}$ stands for the projection in $L^2(\mathbb{R})$ to the subspace $\text{PW}_{[-a, 0]}$. It is easy to see that Γ_ψ is completely determined by its standard symbol $\psi_{st, 2a} = \mathcal{F}^{-1}\chi_{(-2a, 0)}\mathcal{F}\psi$, that is, $\Gamma_\psi f = \Gamma_{\psi_{st, 2a}} f$ for all functions $f \in \text{PW}_{[0, a]}$ such that $\sup_{x \in \mathbb{R}} |xf(x)| < \infty$. Clearly, such functions form a dense subset in $\text{PW}_{[0, a]}$.

It is known that the embedding operator $V_{\mu_a} : \text{PW}_{[0, a]} \rightarrow L^2(\mu_a)$ taking a function $f \in \text{PW}_{[0, a]}$ into its restriction to \mathbb{Z}_a is unitary. The same is true for the embedding operator $\tilde{V}_{\nu_a} : \text{PW}_{[-a, 0]} \rightarrow L^2(\nu_a)$. A general version of the following result is Lemma 4.2 of [1].

Lemma 6.1. *Let $a > 0$, $0 < p \leq 1$, and let $\psi \in L^\infty(\mathbb{Z}_{2a})$. Then there exists an entire function Ψ such that $\Psi = \psi$ on \mathbb{Z}_{2a} , $|F(x)| \leq c \log(e + |x|)$ for all $x \in \mathbb{R}$, and the Fourier spectrum of F is contained in the interval $[-2a, 0]$. Moreover, we have*

$$\tilde{V}_{\nu_a} \Gamma_\Psi V_{\mu_a}^{-1} = -i\tilde{C}_\psi. \quad (7)$$

for the operators $\Gamma_\Psi : \text{PW}_{[0, a]} \rightarrow \text{PW}_{[-a, 0]}$ and $\tilde{C}_\psi : L^2(\mu_a) \rightarrow L^2(\nu_a)$.

Proof. Existence of such a function Ψ follows from a general theory of entire functions, see, e.g., Theorem 1 in Section 21.1 of [2] and Problem 1 after its proof. In order to prove formula (7), take a pair of functions $f \in L^2(\mu_a)$, $g \in L^2(\nu_a)$ with finite support. Consider the functions F, G in $\text{PW}_{[0, a]}$ such that $F = V_{\mu_a}^{-1}f$, $\bar{G} = \tilde{V}_{\nu_a}^{-1}g$. It is easy to see that $\int_{\mathbb{R}} |\Psi FG| dx < \infty$ and hence the bilinear form of Γ_Ψ is correctly defined on functions F, \bar{G} . We have

$$\begin{aligned} (\tilde{V}_{\nu_a} \Gamma_\Psi V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} &= (\Gamma_\Psi F, \bar{G})_{L^2(\mathbb{R})} = (FG, \bar{\Psi})_{L^2(\mathbb{R})} = \\ &= (V_{\mu_{2a}} FG, V_{\mu_{2a}} \bar{\Psi})_{L^2(\mu_{2a})} = \frac{1}{2}(Fg, \bar{\psi})_{L^2(\nu_a)} + \frac{1}{2}(fG, \bar{\psi})_{L^2(\mu_a)}. \end{aligned}$$

For every point $x \in \frac{\pi}{a} + \mathbb{Z}_a$ we have

$$F(x) = (V_{\mu_a} F, V_{\mu_a} k_{x, a})_{L^2(\mu_a)} = \frac{2}{\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - x} d\mu_a(t), \quad x \in \frac{\pi}{a} + \mathbb{Z}_a.$$

Analogously, $G(t) = \frac{2}{\pi i} \int_{\mathbb{R}} \frac{\overline{g(x)}}{x - t} d\nu_a(x)$ for all $t \in \mathbb{Z}_a$. Using these formulas, we get

$$\begin{aligned} (\tilde{V}_{\nu_a} \Gamma_\Psi V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} &= \frac{1}{\pi i} \int_{\mathbb{R}} \frac{\psi(x) - \psi(t)}{x - t} f(t) \overline{g(x)} d\mu_a(t) d\nu_a(x) \\ &= -i(\tilde{C}_\psi f, g)_{L^2(\nu_a)}. \end{aligned}$$

The lemma follows. \square

Proof of Theorem 2'. Let ψ be a function on the lattice \mathbb{Z}_a such that the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ belongs to the class \mathcal{S}^p . Consider the sequence of points $x_k = \frac{2\pi}{a}k$, $k \in \mathbb{Z}$. Since $0 < p \leq 1$, we have

$$\sum_{k \in \mathbb{Z}} |\psi(x_{2k}) - \psi(x_{2k+1})| = \frac{a}{8} \sum_{k \in \mathbb{Z}} |(\tilde{C}_\psi \delta_{x_{2k}}, \delta_{x_{2k+1}})_{L^2(\nu_{\frac{a}{2}})}| < \infty.$$

Hence, the function ψ is bounded on \mathbb{Z}_a . Using Lemma 6.1, we can find an entire function Ψ such that $\Psi = \psi$ on \mathbb{Z}_a , $|\Psi(x)| \leq c \log(e + |x|)$ for all $x \in \mathbb{R}$, the Fourier spectrum of Ψ is contained in $[-a, 0]$, and relation (7) holds for the operators $\Gamma_\Psi : \text{PW}_{[0, \frac{a}{2}]} \rightarrow \text{PW}_{[-\frac{a}{2}, 0]}$ and $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$. In particular, we have $\Gamma_\Psi \in \mathcal{S}^p$. Denote by M the multiplication operator on $L^2(\mathbb{R})$ by the function $e^{\frac{iax}{2}}$. Let $T_{e^{\frac{iax}{2}}\Psi}$ be the Toeplitz operator on $\text{PW}_{\frac{a}{4}}$ with standard symbol $e^{\frac{iax}{2}}\Psi$. Observe that

$$T_{e^{\frac{iax}{2}}\Psi} f = M \Gamma_\Psi M f, \quad (8)$$

for every function $f \in \text{PW}_{\frac{a}{4}}$ such that $\sup_{x \in \mathbb{R}} |xf(x)| < \infty$. Since M maps unitarily $\text{PW}_{\frac{a}{4}}$ onto $\text{PW}_{[0, \frac{a}{2}]}$ and $\text{PW}_{[-\frac{a}{2}, 0]}$ onto $\text{PW}_{\frac{a}{4}}$, the operator $T_{e^{\frac{iax}{2}}\Psi}$ belongs to $\mathcal{S}^p(\text{PW}_{\frac{a}{4}})$. In particular, there exists a function $\varphi \in L^\infty(\mathbb{R})$ such that $T_\varphi = T_{e^{\frac{iax}{2}}\Psi}$ and $\varphi_{st} = e^{\frac{iax}{2}}\Psi + c_1 e^{-i\frac{a}{2}x} + c_2 e^{i\frac{a}{2}x}$ for some constants c_1, c_2 . Since $e^{\frac{iax}{2}}\varphi_{st}$ coincides with $\psi + c_1 + c_2$ on \mathbb{Z}_a , we have $\psi \in \mathbb{B}_p(a, \text{osc})$ by Theorem 1. Moreover, the quasi-norm $\|\tilde{C}_\psi\|_{\mathcal{S}^p}$ is comparable to $\|\psi\|_{\mathbb{B}_p(a, \text{osc})}$ with constants depending only on $p \in (0, 1]$.

Conversely, suppose that $\psi \in \mathbb{B}_p(a, \text{osc}) \cap L^\infty(\mathbb{Z}_a)$. Using Lemma 6.1 again, we find an entire function Ψ such that $\Psi = \psi$ on \mathbb{Z}_a , $|\Psi(x)| \leq c \log(e + |x|)$ for all $x \in \mathbb{R}$, the Fourier spectrum of Ψ is contained in $[-a, 0]$, and relation (7) holds for the operators $\Gamma_\Psi : \text{PW}_{[0, \frac{a}{2}]} \rightarrow \text{PW}_{[-\frac{a}{2}, 0]}$ and $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$. Since $\psi \in L^\infty(\mathbb{Z}_a)$, the operators \tilde{C}_ψ and Γ_Ψ are bounded. Let $\Psi_{st,a}$ be the standard symbol of the operator Γ_Ψ . Note that $\Psi_{st,a}(x) = \Psi(x) + q(x)$ for all $x \in \mathbb{Z}_a$ and a polynomial q of degree at most one. In particular, we have $\Psi_{st,a} \in \mathbb{B}_p(a, \text{osc})$. By Theorem 1, the operator $T_{e^{\frac{iax}{2}}\Psi_{st,a}}$ on $\text{PW}_{\frac{a}{4}}$ is in \mathcal{S}^p , hence $\Gamma_\Psi \in \mathcal{S}^p$ by formula (8). It follows that the operator \tilde{C}_ψ is in \mathcal{S}^p as well, and, moreover, we have the estimate

$$\|\tilde{C}_\psi\|_{\mathcal{S}^p} = \|\Gamma_\Psi\|_{\mathcal{S}^p} = \left\| T_{e^{\frac{iax}{2}}\Psi_{st,a}} \right\|_{\mathcal{S}^p} \leq c_p \|\Psi_{st,a}\|_{\mathbb{B}_p(a, \text{osc})} = c_p \|\psi\|_{\mathbb{B}_p(a, \text{osc})},$$

for a constant c_p depending only on p . The theorem is proved. \square

Proof of Theorem 2. Let ψ be a function on the lattice \mathbb{Z}_a such that we have $C_\psi \in \mathcal{S}^1(L^2(\mu_a))$. Then the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \rightarrow L^2(\nu_{\frac{a}{2}})$ is of trace class as well and $\|\psi\|_{\mathbb{B}_1(a, \text{osc})} \leq c_1 \|\tilde{C}_\psi\|_{\mathcal{S}^1(L^2(\mu_a))} \leq c_1 \|C_\psi\|_{\mathcal{S}^1(L^2(\mu_a))}$ by Theorem 2'.

Conversely, suppose that $\psi \in \mathbb{B}_1(a, \text{osc}) \cap L^\infty(\mathbb{Z}_a)$. By Lemma 4.2, we can find a function $\Psi \in \mathbb{B}_1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\Psi = \psi$ on \mathbb{Z}_a and $\|\Psi\|_{\mathbb{B}_1(\mathbb{R})} \leq c_1 \|\psi\|_{\mathbb{B}_1(a, \text{osc})}$. Denote $\psi_\lambda : t \mapsto \frac{|\text{Im} \lambda|^2}{(t - \lambda)^2}$ for $\lambda \in \mathbb{C}$. Let us apply Theorem 2.10 in [8] to analytic and anti-analytic parts of Ψ : find numbers c, c_λ such that $\sum_{\lambda \in \mathcal{U}_\varepsilon} |c_\lambda| \leq c_1 \|\Psi\|_{\mathbb{B}_1(\mathbb{R})}$ and

$$\psi(x) = \Psi(x) = c + \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda \psi_\lambda(x), \quad x \in \mathbb{Z}_a.$$

We claim that for every $\lambda \in \mathcal{U}_\varepsilon$ the commutator C_{ψ_λ} belongs to the trace class and $\|C_{\psi_\lambda}\|_{\mathcal{S}^1} \leq c_1(1+a)$ for a constant c_1 do not depending on λ . Clearly, this will yield the desired estimate $\|C_\psi\|_{\mathcal{S}^1} \leq c_1(1+a)\|\psi\|_{\mathbb{B}_1(a, \text{osc})}$. We have

$$\frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} = -\frac{|\text{Im } \lambda|^2}{(x - \bar{\lambda})^2(t - \bar{\lambda})} - \frac{|\text{Im } \lambda|^2}{(x - \bar{\lambda})(t - \bar{\lambda})^2}.$$

Denote by K_{ψ_λ} the integral operator on $L^2(\mu_a)$ with kernel $\frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t}$:

$$(K_{\psi_\lambda}f)(x) = \int_{\mathbb{Z}_a} \frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} f(t) dt = (C_{\psi_\lambda}f)(x) + \frac{2|\text{Im } \lambda|^2}{(x - \bar{\lambda})^3} f(x). \quad (9)$$

Observe that the operator K_{ψ_λ} has rank 2 and

$$\|K_{\psi_\lambda}\|_{\mathcal{S}^p} \leq 2|\text{Im } \lambda|^2 \cdot \left\| \frac{1}{(x - \bar{\lambda})^2} \right\|_{L^2(\mu_a)} \left\| \frac{1}{x - \bar{\lambda}} \right\|_{L^2(\mu_a)}.$$

In the case where $\text{dist}(\lambda, \mathbb{Z}_a) \geq \frac{\pi}{2a}$, the last expression could be estimated from above by

$$c_1 \left(\int_{\mathbb{R}} \frac{|\text{Im } \lambda| dt}{t^2 + |\text{Im } \lambda|^2} \int_{\mathbb{R}} \frac{|\text{Im } \lambda|^3 dt}{(t^2 + |\text{Im } \lambda|^2)^2} \right)^{\frac{1}{2}} = c_1 \left(\int_{\mathbb{R}} \frac{dt}{t^2 + 1} \int_{\mathbb{R}} \frac{dt}{(t^2 + 1)^2} \right)^{\frac{1}{2}}.$$

Moreover, the singular numbers of the multiplication operator $f \mapsto \frac{|\text{Im } \lambda|^2}{(x - \bar{\lambda})^3} f$ are precisely $\frac{|\text{Im } \lambda|^2}{|x - \bar{\lambda}|^3}$, $x \in \mathbb{Z}_a$, hence its norm in $\mathcal{S}^1(L^2(\mu_a))$ does not exceed

$$\sum_{x \in \mathbb{Z}_a} \frac{|\text{Im } \lambda|^2}{|x - \bar{\lambda}|^3} \leq \sum_{x \in \mathbb{Z}_a} \frac{|\text{Im } \lambda|^2}{(x^2 + |\text{Im } \lambda|^2)^{\frac{3}{2}}} \leq c_1 a$$

for a universal constant c_1 . This tells us that $\|C_{\psi_\lambda}\|_{\mathcal{S}^p} \leq c_1(1+a)$ for all $\lambda \in \mathcal{U}_\varepsilon$ such that $\text{dist}(\lambda, \mathbb{Z}_a) \geq \frac{\pi}{2a}$. Now consider the case where $\text{dist}(\lambda, \mathbb{Z}_a) \leq \frac{\pi}{2a}$. Let x_λ be the nearest point to λ in the lattice \mathbb{Z}_a . The function ψ_λ belongs to $L^1(\mu_a)$ and

$$\begin{aligned} \sum_{x \in \mathbb{Z}_a} |\psi_\lambda(x)| &\leq |\psi_\lambda(x_\lambda)| + 2|\text{Im } \lambda|^2 \sum_{k=1}^{\infty} \frac{1}{\left(\frac{2\pi}{a}k - \frac{\pi}{2a}\right)^2}, \\ &\leq \left| \frac{\text{Im } \lambda}{\lambda - x_\lambda} \right|^2 + 2 \left(\frac{a|\text{Im } \lambda|}{2\pi} \right)^2 \sum_{k=1}^{\infty} \frac{1}{\left(k - \frac{1}{4}\right)^2} \leq c_1, \end{aligned}$$

where the right hand side does not depend on λ . It follows that the operator M_{ψ_λ} lies in $\mathcal{S}^1(L^2(\mu_a))$ and $\|M_{\psi_\lambda}\|_{\mathcal{S}^1} \leq c_1$. We also have

$$\|C_{\psi_\lambda}\|_{\mathcal{S}^p} = \|H_{\mu_a} M_{\psi_\lambda} - M_{\psi_\lambda} H_{\mu_a}\|_{\mathcal{S}^1} \leq c_1,$$

for another constant c_1 , because the discrete Hilbert transform H_{μ_a} is bounded on $L^2(\mu_a)$. This completes the proof. \square

Remark that the second part of the proof of Theorem 2 is almost literal repetition of the corresponding part of the proof of Theorem 6.2 in [9]. However, the original argument in [9] has a gap: it does not involve the estimate of the \mathcal{S}^1 -norm of the multiplication operator $f \mapsto \frac{|\text{Im } \lambda|^2}{(x - \bar{\lambda})^3} f$ from formula (9). This technical place turns out to be crucial in the case $0 < p < 1$. More precisely, we have the following result.

Proposition 6.1. *Let $0 < p < 1$ and let $a > 0$. There exists a function $\psi \in \mathbb{B}_p(\mathbb{Z}_a)$ such that $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$.*

Proof. Suppose that $C_\psi \in \mathcal{S}^p(L^2(\mu_a))$ for every $\psi \in \mathbb{B}_p(a, \text{osc})$. Then it is easy to see from the closed graph theorem that there exists a constant $c_{p,a}$ such that $\|C_\psi\|_{\mathcal{S}^p} \leq c_{p,a} \|\psi\|_{\mathbb{B}_p(a, \text{osc})}$ for all $\psi \in \mathbb{B}_p(a, \text{osc})$. Take $\lambda \in \mathbb{C}^+$ such that $\text{Im } \lambda \geq \frac{2\pi}{a}$ and consider the function $\psi_\lambda : t \mapsto \frac{\text{Im } \lambda}{t - \bar{\lambda}}$. Analogously to (9), we have $K_{\psi_\lambda} = C_{\psi_\lambda} + M_\lambda$, where K_{ψ_λ} is the integral operator with kernel

$$\frac{\psi_\lambda(x) - \psi_\lambda(t)}{x - t} = -\frac{\text{Im } \lambda}{(x - \bar{\lambda})(t - \bar{\lambda})},$$

and $M_\lambda : f \mapsto \frac{\text{Im } \lambda}{(x - \bar{\lambda})^2} f$ is the multiplication operator on $L^2(\mu_a)$ by $\frac{\text{Im } \lambda}{(x - \bar{\lambda})^2}$. Observe that K_{ψ_λ} is the rank-one operator whose norm does not exceed

$$\text{Im } \lambda \cdot \left\| \frac{1}{x - \bar{\lambda}} \right\|_{L^2(\mu_a)}^2 \leq c_p \int_{\mathbb{R}} \frac{\text{Im } \lambda \, dt}{t^2 + (\text{Im } \lambda)^2} = c_p \int_{\mathbb{R}} \frac{dt}{t^2 + 1}.$$

It follows from our assumption and Lemma 4.1 that $\|M_\lambda\|_{\mathcal{S}^p} \leq c_{p,a}$ for all $\lambda \in \mathbb{C}$ with $\text{Im } \lambda \geq \frac{2\pi}{a}$ and a universal constant c_p . On the other hand, we have

$$\|M_\lambda\|_{\mathcal{S}^p}^p = \sum_{x \in \mathbb{Z}_a} \frac{(\text{Im } \lambda)^p}{|x - \bar{\lambda}|^{2p}} \geq ac_p \int_{\mathbb{R}} \frac{(\text{Im } \lambda)^p \, dx}{(x^2 + (\text{Im } \lambda)^2)^p} = ac_p (\text{Im } \lambda)^{1-p} \int_{\mathbb{R}} \frac{dt}{t^2 + 1}.$$

Since the right hand side is unbounded in λ , we get the contradiction. \square

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